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HYDROELASTICITY OF THE KIRCHHOFF ROD: BUCKLING PHENOMENA

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We consider the nonlinear coupled hydroelastic problem of a general curved and twisted flexible slender structure (i.e. flexible riser, cable system, fish–farm net system, towed arrays, etc.) embedded in a nonuniform flow field such as the ocean environment; the flow direction is arbitrary, relative to the axis of the slender structure. The motion of the elastic structure is coupled with the hydrodynamic loads acting on the slender structure by the ambient flow field. An important input for such hydroelastic problems is the computation of the hydrodynamic loading per unit length experienced by the slender body. A rigorously derived improvement for the inertial loading per unit length over the commonly used Morison-type semi-empirical force (originally obtained for straight long structures in a uniform stream) is used. The structure is also allowed to undergo small (yet finite) deflections from its original reference central-line, due to a particular model of intrinsic elasticity governed by a corresponding *nonlinear* PDE, which corresponds to the well-known Kirchhoff rod elastic model. The system of coupled hydroelastic equations is investigated in order to derive analytically the influence of the hydrodynamic loading in a uniform stationary stream on the nonlinear stability of the straight rod. It is found that the presence of an ambient stationary stream decreases the critical parameters (critical twist) of the buckling phenomenon which is known to exist for the same rod when placed in a vacuum. Also revealed is a new type of stability loss, which is affected by viscous effects.
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1. INTRODUCTION

The study of slender elastic structures is a subject of continued scientific and mathematical interest. The problem is much more difficult when the structure is assumed to be embedded in an ambient stream of a heavy liquid with a certain rheology. Among the most important hydroelastic applications one can mention, for example, the writhing and buckling of submerged mooring and telephone cables (Coyne 1990), and the stability of floating structures such as offshore platforms, ships and buoys in the ocean [e.g. Zhu *et al.* (1999)], etc.

The history of the study of slender elastic structures started with the paper of Kirchhoff, later extended by Love (1927). They studied the so-called Kirchhoff rod model, in which a twist and stretching of the structure are both included in the formulation in a direct way. Their classical approach is based on the so-called "kinetic analogue" (Love 1927), and mainly treats the problem of the classification and stability of equilibrium positions of structures *in vacuo* (Goriely & Tabor 1997).

In order to treat hydroelastic problems involving slender structures, one can introduce the so-called "hydrodynamic line" limit (Rainey 1995; Galper *et al.* 1996). In this limit the structure is considered as an effectively 1-D structure embedded in a 3-D liquid stream. Correspondingly, the intrinsic differential geometry of this 1-D shape should be connected with the flow description of the ambient stream. This poses a nontrivial problem of

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determining the cross-sectional hydrodynamic loading acting on a slender deformable structure in the limit of the hydrodynamic line (Galper & Miloh 1999).

Traditionally (Zhu *et al.* 1999), the corresponding expression for the loading has been used in a partly empirical form based on the so-called "Morison formula" (Sarpkaya & Isaacson 1981; Lighthill 1986). The Morison expression consists of two parts accounting for the inertial (potential) forcing and viscous drag, respectively. An exact expression for the cross-sectional loading in a potential stream has only recently been derived by Galper & Miloh (1999). This derivation enables us to conduct a rigorous analytical treatment of the hydroelastic problem of a slender structure for the case of an inviscid stream. For an arbitrary, viscous, nonuniform ambient flow field, an exact solution is not available.

As a next step, one can then add the hydrodynamic loading to the right-hand side of the corresponding linear (or, in general, nonlinear) hydroelastic equations governing the dynamics of the structure immersed in a fluid. In this way, and even for a linear elastic model, one obtains a system of nonlinear PDEs with a distribution of contact forces.

We choose the Kirchhoff rod to model the dynamics of immersed cables. This model is fully consistent with the above-mentioned hydrodynamic-line limit, in the sense that it is of the same order in the characteristic small parameter (namely, the slenderness of the structure).

In this paper we investigate the stability and buckling phenomenon of a *straight* rod embedded in a *uniform* ambient stream. It is well known that a corresponding buckling phenomenon occurs for a Kirchhoff rod *in vacuo* (Love 1927). The straight rod bifurcates into a helical configuration, and helical equilibrium configurations lose their stability by means of coiling. The stability problem of various types of buckling is considered in a number of recent papers of Goriely & Tabor (1997, 1998), treating mainly the linear stability of some equilibrium configurations based on the neutral curve consideration. Following this methodology, we determine the influence of a uniform stream on the critical bifurcation parameters for the buckling of a straight Kirchhoff rod. We find that the stream decreases the corresponding critical parameters of the bifurcation, when compared with the corresponding values *in vacuo*, leading to a destabilization of the rod.

The paper is organized as follows: In Sections 2 and 3 we set up the generalized equations which govern the dynamics of a Kirchhoff rod lying in an ambient arbitrary stream. The flow–structure interaction problem is treated within a potential flow framework. In Section 4 we consider a special case of a *uniform* ambient stream. The corresponding linear stability perturbation technique is next developed in Sections 5 and 6. It is then applied to the case of a straight rod which is subject to stretching and a constant twist. Exact expressions for the new critical bifurcation parameters are established and the influence of a uniform viscous drag on the rod stability is also discussed in Section 7.

2. A KIRCHHOFF ROD *IN VACUO*

We choose to model a cable by a slender elastic rod which satisfies the so-called Kirchhoff rod theory [see Coleman *et al.* (1993)]. According to this theory the strains are assumed small, when compared with the undisturbed configuration, and so are the nondimensional thickness, curvature and twist. Thus, the Kirchhoff theory is correct up to the second order in these parameters. The rod is considered to be inextensible, which is in full correspondence with the first-order theory. Kinematically speaking, this model captures in a fairly good manner the essential long-wave properties of the elastic structure.

The rod L, of a circular cross-section $S: r - a = 0$, is specified by its curved time-dependent smooth central-line $C(t)$. This centreline can be represented by the position vector $\mathbf{X}(s,t)$ of a point on $C(s,t)$, whereas s is the natural parameter (arc-length) of the inextensible rod. Thus, one can introduce a unit vector $\mathbf{\Theta}(s,t)$ tangent to the curve $C(s,t)$ given by

$$\frac{\partial \mathbf{X}(s,t)}{\partial s} = \mathbf{\Theta}(s,t), \quad |\mathbf{\Theta}| = 1. \tag{1}$$

We also choose two arbitrary unit mutually orthogonal vectors $\mathbf{d}_1(s,t)$ and $\mathbf{d}_2(s,t)$ in such a way that the triad $\mathbf{\Theta}$, $\mathbf{d}_1$, $\mathbf{d}_2$ forms a right-handed orthonormal frame along $C(s,t)$. One can consider the triad (for a fixed t) as if it were a rigid body, rotating along the central curve. In this case, one can interpret s as time which is the nature of the well-known Kirchhoff kinetic analogue (Coleman *et al.* 1993). This "rotation" can be described with the help of the "Darboux vector" $\mathbf{\Omega}$ [see, for example, Section 5 of Dubrovin *et al.* (1984)], where $\mathbf{\Omega} = (\kappa_1,\ \kappa_2,\ \tau)$, κ_1 and κ_2 representing the curvatures of the centre line when projected onto planes normal to $\mathbf{d}_1, \mathbf{d}_2$, respectively, and τ representing the twist. Thus, one obtains

$$\frac{\partial \mathbf{\Theta}}{\partial s} + \mathbf{\Omega} \wedge \mathbf{\Theta} = \mathbf{0}, \quad \frac{\partial \mathbf{d}_1}{\partial s} + \mathbf{\Omega} \wedge \mathbf{d}_1 = \mathbf{0}, \quad \frac{\partial \mathbf{d}_2}{\partial s} + \mathbf{\Omega} \wedge \mathbf{d}_2 = \mathbf{0}. \tag{2}$$

If one denotes by $\mathbf{f}(s,t)$ and $\mathbf{m}(s,t)$ the density of the *contact* internal force and moment per unit length, respectively (acting in a direction orthogonal to $\mathbf{\Theta}(s,t)$), the Kirchhoff equations obtain the conservation form for both linear and angular momentum, i.e.

$$\frac{\partial \mathbf{f}(s)}{\partial s} = \rho_b \int_S \frac{\partial^2 \tilde{\mathbf{X}}}{\partial t^2}\, \mathrm{d}S, \tag{3}$$

$$\frac{\partial \mathbf{m}(s)}{\partial s} + \mathbf{\Theta} \wedge \mathbf{f} = \rho_b \int_S \mathbf{r} \wedge \frac{\partial^2 \tilde{\mathbf{X}}}{\partial t^2}\, \mathrm{d}S, \tag{4}$$

where $\mathbf{r}(s,\ t)$ denotes the position vector of a point on the cross-section with respect to the axis, $\tilde{\mathbf{X}} \equiv \mathbf{X} + \mathbf{r}$ and ρ_b denotes the density of the rod.

Equations (3) and (4) should also be augmented by a proper constitutive relationship connecting the moment distribution $\mathbf{m}(s,t)$ with the Darboux vector $\mathbf{\Omega}$ of the system. When we invoke this additional relationship, equations (3) and (4) form a system of nine equations with nine unknowns, namely the Darboux vector, as well as the force $\mathbf{f}(s,t)$ and moment distributions $\mathbf{m}(s,t)$ (Goriely & Tabor 1997).

3. FLUID – STRUCTURE INTERACTION

Let a cylindrical structure with a central-line $C(s,t)$ be placed in a *nonuniform* ambient unsteady flow field $\mathbf{V}(\mathbf{X},t) = \mathbf{\nabla}\phi(\mathbf{X},t)$ with a constant fluid density ρ_f. The corresponding expressions for the hydrodynamic force and moment exerted on the deformable cylinder are given below, in a moving (body-fixed) coordinate system with an origin coinciding instantaneously with its centre of mass.

We denote the deformation velocity of a point s on $C(s,t)$ by $\mathbf{U}^{(d)}(s,t)$. Clearly, for a structure with a fixed centroid location and fixed directions of its main axes

$$\mathbf{U}^{(d)}(s,t) = \frac{\partial \mathbf{X}(s,t)}{\partial t}. \tag{5}$$

Correspondingly, the deformation velocity $V^{(d)}(s,\mathbf{r},\ t)$ of a point $(s,\mathbf{r})$ on the surface L (Galper & Miloh 1995) is given by

$$V^{(d)}(s,\mathbf{r},\ t) = \mathbf{n}(s,\mathbf{r},\ t)\cdot \mathbf{U}^{(d)}(s,\ t), \tag{6}$$

where $\mathbf{n}(s,\mathbf{r},\ t)$ is the normal to the deformable surface L.

One can introduce now the hydrodynamic loading *per unit length*, $\tilde{\mathbf{F}}(s,t)$, given by

$$\mathbf{F}(t) = \int_{H}^{-H} \tilde{\mathbf{F}}(s,t)\,\mathrm{d}s, \tag{7}$$

where $\tilde{\mathbf{F}}(s,t)$ is aligned in a direction orthogonal to $\mathbf{\Theta}(s,t)$ and $\mathbf{F}(t)$ is the total hydrodynamic force acting on the structure with a total length $2H$.

The dynamic hydroelastic Kirchhoff equations replacing equations (3) and (4) which account for the presence of an ambient flow field should be augmented now, by including a corresponding hydrodynamic loading per unit length, i.e.

$$\frac{\partial \mathbf{f}(s)}{\partial s} + \tilde{\mathbf{F}}(s,t) = \rho_b \int_{S(s)} \frac{\partial^2 \tilde{\mathbf{X}}}{\partial t^2}\,\mathrm{d}S, \tag{8}$$

$$\frac{\partial \mathbf{m}(s)}{\partial s} + \mathbf{\Theta} \wedge \mathbf{f} = \rho_b \int_{S(s)} \mathbf{r} \wedge \frac{\partial^2 \tilde{\mathbf{X}}}{\partial t^2}\,\mathrm{d}S. \tag{9}$$

Note that the equation of conservation of angular momentum, equation (9), is the same as equation (4) and that the hydrodynamic loading enters into the formalism only through the linear momentum equation (8).

Consider next the case where the characteristic-length scale l of the nonuniformity of the ambient flow field, $\mathbf{V}$, is much larger than the characteristic length scale $|S|$ of the cylinder cross-section (the so-called “weakly nonuniform” field approximation). In this case, there exists a small parameter which has the sense of a ratio between these two scales, namely

$$\varepsilon \sim \frac{|S|}{l} \ll 1. \tag{10}$$

Furthermore, we assume that the curvature of the structure is small enough (i.e. $\max_s \kappa(s)a = \mathcal{O}(\varepsilon)$), which corresponds to the general limiting procedure of the “hydrodynamic line”, where all parameters are considered as fixed as the cross-section radius a tends to zero (Rainey 1995). Note that we do not impose here any restrictions on the torsion and on the s-derivatives of either curvature or torsion of the structure. The hydrodynamic loading per unit length is calculated to the leading order in the small parameter ε. Such an approximation is fully consistent with the first-order approximation used to derive the elastic model of the Kirchhoff rod.

The hydrodynamic loading $\tilde{\mathbf{F}}(s,t)$ can be split into three parts, namely

$$\tilde{\mathbf{F}}(s,t) = \mathbf{F}(s,t) + \mathbf{F}^{(d)}(s,t) + \mathbf{F}^{(q)}(s,t). \tag{11}$$

Here $\mathbf{F}(s,t)$ is the hydrodynamic loading on the “frozen” rod, imposed by the ambient stream. The term $\mathbf{F}^{(d)}(s,t)$ results from the interaction between the pure deformation of the rod with a nonzero ambient flow field. Finally, $\mathbf{F}^{(q)}(s,t)$ represents the cross-sectional loading due to a pure deformation in a *quiescent* fluid.

One can specify the following expression for the hydrodynamic loading (Galper & Miloh 1999) in terms of the acceleration of the ambient flow field $\mathbf{a} \equiv D\mathbf{V}/Dt$:

$$\mathbf{F}(s)|_T = 2\rho_f \sigma \mathbf{a}|_T - \rho_f \sigma V_\Theta \left.\frac{\partial \mathbf{V}}{\partial s}\right|_T + \rho_f \sigma \frac{\partial}{\partial s}\left.\left(V_\Theta \mathbf{V} - \frac{1}{2}(|\mathbf{V}|^2 + V_\Theta^2)\mathbf{\Theta}\right)\right|_T + \mathcal{O}(\varepsilon^2 \log \varepsilon). \tag{12}$$

It can be shown that the next order terms on the right-hand side of equation (12) are of $\mathcal{O}(\varepsilon^2 \log \varepsilon)$ (Galper *et al.* 1996). Here the substantial time derivative operator is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_\alpha \nabla_\alpha + V_\Theta \frac{\partial}{\partial s} + \mathcal{O}(\varepsilon^2). \tag{13}$$

Further, σ denotes the cross-sectional area, the operator $|_T$ denotes the projection of a function on the corresponding cross-section plane, and we imply a summation over $\alpha = 1, 2$ in equation (13). Equation (12) reduces exactly to the expressions presented in Rainey (1995) for the force distribution acting on a *straight* cylinder (i.e. for $\mathbf{\Omega} = 0$).

It can be also shown in a similar manner (Galper & Miloh 1999) that the additional loading on a *curved* slender structure, due to *deformation*, can be written as

$$\mathbf{F}^{(d)}(s,t) = \rho_f \sigma\, U^{(d)}_{\Theta} \left.\frac{\partial \mathbf{V}}{\partial s}\right|_T - \rho_f \sigma \left.\frac{\partial}{\partial s}\left(V_{\Theta}(s)\mathbf{U}^{(d)}(s)\right)\right|_T + \rho_f \sigma (\mathbf{U}^{(d)}(s)\cdot \mathbf{V}(s)) \frac{\partial \mathbf{\Theta}}{\partial s}. \tag{14}$$

We recall that, according to the Kirchhoff rod model, the cross-section stays orthogonal to the tangential vector, which implies that $\mathbf{U}^{(d)}(s)\cdot\mathbf{\Theta}(s) = 0$.

We emphasize here the *local* character of the loading per unit length in the case of a curved structure with a *constant* cross-section. In this case, the cross-section loading depends only on variables related to the same cross-section. Note that for a slender structure with a *variable* cross-section the corresponding expressions for the loading are generally *nonlocal* (Galper & Miloh 2000).

In order to estimate the effect of viscous flow separation for a cable (modelled as a Kirchhoff rod) in a marine environment, one can calculate the corresponding viscous drag distribution $\mathbf{F}^{(v)}(s,t)$ using the simple Morison formula (Zhu *et al.* 1999). It is given by

$$\mathbf{F}^{(v)}(s,t) = -\tfrac{1}{2}\sigma\rho_f|\mathbf{u}(s,t)|(C_N \mathbf{u}(s,t)|_T + C_{\Theta}(\mathbf{u}(s,t)\cdot\mathbf{\Theta})\mathbf{\Theta}), \tag{15}$$

where $\mathbf{u}(s,t) \equiv (\mathbf{V}(s,t) - \mathbf{U}^{(d)}(s,t))$ represents the relative velocity of the cross-section, and C_N, C_{Θ} are the corresponding normal and tangential drag coefficients, respectively. Note that within the *inextensible* Kirchhoff rod model only the normal loadings are accounted for and hence only the normal part of equation (15) contributes to the force loading.

Finally, using equations (8) and (9), where the integrals on the right-hand side are calculated up to the leading order, we obtain the following generalized hydro-elastic Kirchhoff equations of motion (after taking the s-derivative of equation (8) and using equation (1)):

$$\frac{\partial^2}{\partial s^2}\mathbf{f}(s) + \frac{\partial}{\partial s}(\mathbf{F}|_T(s) + \mathbf{F}^{(d)}|_T(s) + \mathbf{F}^{(v)}|_T(s)) = \sigma(\rho_b + \rho_f)\frac{\partial^2 \mathbf{\Theta}}{\partial t^2}, \tag{16}$$

$$\frac{\partial \mathbf{m}(s)}{\partial s} + \mathbf{\Theta}\wedge\mathbf{f} = \rho_b I\left(\mathbf{d}_1 \wedge \frac{\partial^2 \mathbf{d}_1}{\partial t^2} + \mathbf{d}_2 \wedge \frac{\partial^2 \mathbf{d}_2}{\partial t^2}\right), \tag{17}$$

where $I = \pi a^4/2$ denotes the moment of inertia of a rod cross-section about a centred axis in the plane of the cross-section and the various force loadings $\mathbf{F}(s)$, $\mathbf{F}^{(d)}(s)$ and $\mathbf{F}^{(v)}(s)$ are given by equations (12), (14) and (15), respectively. It is also remarked that the additional term $\sigma\rho_f\partial^2\mathbf{\Theta}/\partial t^2$ on the right-hand side of equation (16), which represents the cross-sectional added-mass, arises from the hydrodynamic loading term $\mathbf{F}^{(q)}$ acting on the deformable structure in a *quiescent* fluid.

For a nonuniform flow field the hydrodynamic loading depends directly on $\mathbf{X}(s,t)$ through the terms $\mathbf{V}(\mathbf{X}(s,t))$. This fact increases the order of the generalized hydro-elastic Kirchhoff equations in comparison with equations (4) and (11). These equations are defined up to rigid-body motions in space and are, therefore, independent on $\mathbf{X}(s,t)$. Instead, they depend on s- and t-derivatives of $\mathbf{X}(s,t)$. Only in the case of a *uniform* flow-field are the generalized Kirchhoff equations (16) and (17) of the same order as equations (4) and (11). When these equations are augmented by a constitutive relationship (see below), they form a *closed* system of nonlinear PDEs.

As shown in Section 4, the additional term on the left-hand side of equation (16) leads to a destabilization of the stationary (equilibrium configurations) solutions of equations (16) and (17). Thus, for example, an equilibrium rod configuration embedded in an ambient (even uniform) flow field tends towards spatial chaos (Coleman *et al.* 1993). Also, a straight elastic rod placed in a constant (or even nonuniform) stream will experience a first bifurcation of its shape at a smaller critical value of twisting moment then the corresponding value *in vacuo*.

4. UNIFORM FLOW-FIELD

Let us consider first the case of a rod embedded in a *stationary, uniform* flow-field $\mathbf{V} =$ constant. For this special case, equation (12) is replaced by

$$\mathbf{F}_{\text{uni}}(s) = \rho_f \sigma \frac{\partial (V_\Theta \mathbf{V} - \frac{1}{2}(|V|^2 + V_\Theta^2)\boldsymbol{\Theta})}{\partial s}. \tag{18}$$

Noting further that $\mathbf{F}_{\text{uni}} \cdot \boldsymbol{\Theta} = 0$, we conclude that

$$\mathbf{F}_{\text{uni}}(s)|_T = \mathbf{F}_{\text{uni}}(s). \tag{19}$$

Hence,

$$\frac{\partial(\mathbf{F}_{\text{uni}}|_T)}{\partial s} = \rho_f \sigma \frac{\partial^2 (V_\Theta \mathbf{V} - \frac{1}{2}(|V|^2 + V_\Theta^2)\boldsymbol{\Theta})}{\partial s^2}. \tag{20}$$

The deformation loading given by equation (14) is next simplified for a stationary uniform flow field to

$$\begin{aligned}\mathbf{F}^{(d)}_{\text{uni}}(s) &= -\rho_f \sigma \left(\frac{\partial (V_\Theta \mathbf{U}^{(d)})}{\partial s} - (\mathbf{U}^{(d)} \cdot \mathbf{V}) \frac{\partial \boldsymbol{\Theta}}{\partial s} \right) \\ &= -\rho_f \sigma \frac{\partial (V_\Theta \mathbf{U}^{(d)} - (\mathbf{U}^{(d)} \cdot \mathbf{V})\boldsymbol{\Theta})}{\partial s} - \rho_f \sigma \left(\frac{\partial \boldsymbol{\Theta}}{\partial t} \cdot \mathbf{V} \right) \boldsymbol{\Theta}.\end{aligned} \tag{21}$$

Here also $\mathbf{F}^{(d)}_{\text{uni}}(s)|_T = \mathbf{F}^{(d)}_{\text{uni}}(s)$ and thus

$$\frac{\partial \mathbf{F}^{(d)}_{\text{uni}}(s)}{\partial s} = -\rho_f \sigma \frac{\partial^2 (V_\Theta \mathbf{U}^{(d)} - (\mathbf{U}^{(d)} \cdot \mathbf{V})\boldsymbol{\Theta})}{\partial s^2} - \rho_f \sigma \frac{\partial}{\partial s} \left[\left(\frac{\partial \boldsymbol{\Theta}}{\partial t} \cdot \mathbf{V} \right) \boldsymbol{\Theta} \right]. \tag{22}$$

It is convenient at this stage to introduce the following scaling:

$$s \to \sqrt{\frac{I}{\sigma}}\, s, \quad t \to \sqrt{\frac{I}{\sigma}} \frac{1}{V_e}\, t, \quad \mathbf{f} \to \sigma E \mathbf{f}, \quad \mathbf{V} \to V_e\, \mathbf{V}, \quad \mathbf{m} \to \sqrt{\sigma I} E \mathbf{m}, \tag{23}$$

where E is the Young's modulus, and the elastic wave velocity is defined as

$$V_e \equiv \sqrt{E/\rho_b}. \tag{24}$$

Generally, a typical dimensionless velocity in the ocean satisfies $V \ll 1$.

The generalized Kirchhoff equations for an elastic rod in a uniform stationary stream can be now expressed in terms of these dimensionless variables as

$$\frac{\partial^2 \mathbf{f}(s)}{\partial s^2} + \mu \frac{\partial^2 (V_\Theta \mathbf{V} - \frac{1}{2}(|V|^2 + V_\Theta^2)\boldsymbol{\Theta})}{\partial s^2} + \frac{\partial (\mathbf{F}^{(d)}_{\text{uni}} + \mathbf{F}^{(v)})}{\partial s} = (1 + \mu) \frac{\partial^2 \boldsymbol{\Theta}}{\partial t^2}, \tag{25}$$

$$\frac{\partial \mathbf{m}(s)}{\partial s} + \boldsymbol{\Theta} \wedge \mathbf{f} = \left(\mathbf{d}_1 \wedge \frac{\partial^2 \mathbf{d}_1}{\partial t^2} + \mathbf{d}_2 \wedge \frac{\partial^2 \mathbf{d}_2}{\partial t^2} \right), \tag{26}$$

where $\mathbf{F}_{\text{uni}}^{(d)}$ is given by the dimensionless form of equation (21) and $\mu \equiv \rho_f/\rho_b$. Complementing equations (25) and (26), we introduce the following linear constitutive relationship:

$$\mathbf{m} = \Omega_1 \mathbf{d}_1 + \Omega_2 \mathbf{d}_2 + \Gamma \Omega_3 \mathbf{\Theta}, \tag{27}$$

where Γ is a certain constant ($\frac{2}{3} \leq \Gamma \leq 1$) depending on the rod elasticity (Goriely & Tabor 1997). We then obtain a closed system of differential equations.

It is known that a "dry" rod has a number of stationary equilibrium configurations *in vacuo*, among which the simplest ones are straight, circular and helicoidal configurations [see Nizette & Goriely (1999) for a detailed classification]. Consider for example a *stretched* straight rod which is subject to a constant density twist γ and a constant dimensionless tension P^2 acting along the rod (defined as a squared quantity to emphasize its positive nature). This can be described in our dynamic variables by

$$\mathbf{f}^{(0)} = (0, 0, P^2), \quad \mathbf{\Omega}^{(0)} = (0, 0, \gamma), \tag{28}$$

which has been shown to be a solution of equations (25)–(27). Indeed, the cross-sectional inertial loading on a *straight* rod placed in a uniform flow field is equal to zero, whereas the viscous force is constant and thus has a vanishing s-derivative.

5. THE BUCKLING PHENOMENON

We now apply the generalized Kirchhoff equations (25)–(27) to analyze the buckling problem of a straight rod which is subject to the constant tension and twist given by equation (28). It is well-known (Goriely & Tabor 1997) that the first bifurcation from a straight to a helicoidal shape *in vacuo*, occurs when (for, say, fixed P)

$$\gamma \geq \gamma_{\text{cr}} = \frac{2P}{\Gamma}. \tag{29}$$

The stability of all possible equilibrium positions can be considered along two complementary lines. First, the stability analysis can be based on a *stationary* system of the governing differential equations. A more consistent way is to take into consideration the full dynamic description. In the latter case, the bifurcation criteria can be obtained from the standard linear stability analysis, by deriving the corresponding neutral stability curve. In this derivation, the terms which are proportional to a time-derivative of the variables do not contribute, and hence the deformational loading term in equation (25) does not affect the linear stability analysis.

Let us introduce further the following new variable;

$$\tilde{\mathbf{f}} \equiv \mathbf{f} - \frac{\mu}{2}(|\mathbf{V}|^2 + V_\Theta^2)\,\mathbf{\Theta} \tag{30}$$

through which the conservation statements can be written as

$$\frac{\partial^2 \tilde{\mathbf{f}}(s)}{\partial s^2} + \mu \frac{\partial^2 (V_\Theta \mathbf{V})}{\partial s^2} + \frac{\partial(\mathbf{F}_{\text{uni}}^{(d)} + \mathbf{F}^{(v)})}{\partial s} = (1+\mu)\frac{\partial^2 \mathbf{\Theta}}{\partial t^2}, \tag{31}$$

$$\frac{\partial \mathbf{m}(s)}{\partial s} + \mathbf{\Theta} \wedge \tilde{\mathbf{f}} = \left(\mathbf{d}_1 \wedge \frac{\partial^2 \mathbf{d}_1}{\partial t^2} + \mathbf{d}_2 \wedge \frac{\partial^2 \mathbf{d}_2}{\partial t^2}\right). \tag{32}$$

Equation (27) remains unchanged. The initial tension in the rod is now written as

$$\tilde{\mathbf{f}}^{(0)} = \left(0, 0, P^2 - \frac{\mu}{2}(|V|^2 + V_\Theta^2)\right) \equiv (0, 0, R). \tag{33}$$

Neglecting all terms with time-derivatives in equations (31), (32) and (27), we finally obtain the following governing equations for the equilibrium configuration:

$$\frac{\partial^2 \tilde{\mathbf{f}}(s)}{\partial s^2} + \mu \frac{\partial^2 (V_\Theta \mathbf{V})}{\partial s^2} + \mu C_L |V|^2 \frac{\partial \boldsymbol{\Theta}}{\partial s} = \mathbf{0}, \tag{34}$$

$$\frac{\partial \mathbf{m}(s)}{\partial s} + \boldsymbol{\Theta} \wedge \tilde{\mathbf{f}} = \mathbf{0}, \tag{35}$$

where $C_L \equiv \frac{1}{2}\sqrt{\frac{I}{\sigma}} C_N$. Equations (34) and (35) are supplemented by the constitutive relationship (27). Clearly, a straight configuration, which is given by $\boldsymbol{\Omega}^{(0)} = (0, 0, \gamma)$ with $\tilde{\mathbf{f}}^{(0)}$ defined in equation (33), is a solution of equations (34), (35) and (27).

6. LINEAR STABILITY: INVISCID CASE

Consider further the linear stability problem in the case where the viscous term in the Morison formula can be neglected in comparison with the inertial loading term. In order to construct an appropriate linear stability analysis, we will use a perturbation technique which was recently developed by Goriely & Tabor (1997). Thus, one can introduce

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}^{(0)} + \varepsilon(\boldsymbol{\Phi} \wedge \boldsymbol{\Theta}^{(0)}) + \mathcal{O}(\varepsilon^2), \quad \mathbf{d}_i = \mathbf{d}_i^{(0)} + \varepsilon(\boldsymbol{\Phi} \wedge \mathbf{d}_i^{(0)}) + \mathcal{O}(\varepsilon^2), \quad i = 1, 2, \tag{36}$$

where ε is a small parameter used in the perturbation procedure and $\mathbf{d}_1^{(0)}$, $\mathbf{d}_2^{(0)}$ and $\boldsymbol{\Theta}^{(0)}$ are the triad vectors of the undisturbed straight configuration. The vector $\boldsymbol{\Phi}$ appearing in equation (36) has the sense of an angular velocity of a rigid triad $(\mathbf{d}_1^{(0)}, \mathbf{d}_2^{(0)}, \boldsymbol{\Theta}^{(0)})$ due to the perturbation. In a similar manner, let us define

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}^{(0)} + \varepsilon\left(\frac{\partial \boldsymbol{\Phi}}{\partial s} + \boldsymbol{\Omega}^{(0)} \wedge \boldsymbol{\Phi}\right), \quad \tilde{\mathbf{f}} = \tilde{\mathbf{f}}^{(0)} + \varepsilon(\tilde{\mathbf{f}}^{(1)} + \boldsymbol{\Phi} \wedge \tilde{\mathbf{f}}^{(0)}) + \mathcal{O}(\varepsilon^2). \tag{37}$$

By introducing equations (36) and (37) into equations (31), (35) and (27), and using equation (28), we search for real linear solutions of $\boldsymbol{\Phi}$ and $\tilde{\mathbf{f}}^{(1)}$ in the form

$$\Phi_i = \mathrm{e}^{\zeta t}(Ax_i \mathrm{e}^{\mathrm{i}\omega s} + \text{c.c.}), \quad \tilde{f}_i^{(1)} = \mathrm{e}^{\zeta t}(Ay_i \mathrm{e}^{\mathrm{i}\omega s} + \text{c.c.}) \quad \text{for} \quad i = 1, 2, 3. \tag{38}$$

After some tedious calculations, one obtains the following expression for the case $\zeta = 0$, (determining the neutral curve in our linear stability problem) for the variables $\boldsymbol{\Phi}$ and $\tilde{\mathbf{f}}^{(1)}$ resulting from equation (31):

$$\begin{aligned} &(\omega^2 + \gamma^2)\tilde{f}_1^{(1)} + 2\mathrm{i}\gamma\omega \tilde{f}_2^{(1)} - 2\mathrm{i}\gamma\omega R^2 \Phi_1 + (\omega^2 + \gamma^2)\tilde{R}^2 \Phi_2 = 0, \\ &2\mathrm{i}\gamma\omega \tilde{f}_1^{(1)} - (\omega^2 + \gamma^2)\tilde{f}_2^{(1)} + (\omega^2 + \gamma^2) R^2 \Phi_1 + 2\mathrm{i}\gamma\omega \tilde{R}^2 \Phi_2 = 0, \end{aligned} \tag{39}$$

where

$$\tilde{R}^2 \equiv R^2 + \mu |V|^2 \sin^2 v, \tag{40}$$

and R is defined in equation (33). Here the angle v is the angle between the uniform stream and the undisturbed straight rod, i.e. $\mathbf{V} \cdot \boldsymbol{\Theta}^{(0)} = |V| \cos v$. Equation (35) also leads to

$$\begin{aligned} &\tilde{f}_1^{(1)} - \mathrm{i}(\Gamma - 2)\gamma\omega \Phi_1 - (\omega^2 + (1 - \Gamma)\gamma^2)\Phi_2 = 0, \\ &\tilde{f}_2^{(1)} + (\omega^2 + (1 - \Gamma)\gamma^2)\Phi_1 - \mathrm{i}(\Gamma - 2)\gamma\omega \Phi_2 = 0. \end{aligned} \tag{41}$$

The neutral curve is given next by the vanishing condition of the corresponding determinant, resulting from equations (39) and (41). Thus

$$\begin{vmatrix} 1 & 0 & -\theta & \beta \\ 0 & 1 & \beta & \theta \\ \eta & 2\mathrm{i}\gamma\omega & -2\mathrm{i}\gamma R^2 & \eta\tilde{R}^2 \\ 2\mathrm{i}\gamma\omega & -\eta & \eta R^2 & 2\mathrm{i}\gamma\tilde{R}^2\omega \end{vmatrix} = 0, \tag{42}$$

where we introduce the notation

$$\eta \equiv (\omega^2 + \gamma^2), \quad \theta \equiv \mathrm{i}(\Gamma - 2)\,\gamma\omega, \qquad \beta \equiv (\omega^2 + (1 - \Gamma)\,\gamma^2). \tag{43}$$

To calculate the determinant $\Delta(\omega)$ given in equation (42), we use the following formula:

$$\det\begin{vmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{vmatrix} = \det(\hat{A})\det(\hat{D} - \hat{C}\hat{B}). \tag{44}$$

Thus, using equation (44) one obtains for the neutral curve $\Delta(\omega)$:

$$\Delta(\omega) = (\omega^2 - \gamma^2)^2[\gamma^2(\Gamma - 1) - R^2 - \omega^2][\gamma^2(\Gamma - 1) - \tilde{R}^2 - \omega^2] = 0. \tag{45}$$

The first root of equation (45), given by $\omega = \gamma$, does not change the straight configuration (due to the fact that $\mathbf{\Omega}^{(1)} = 0$ for $\omega = \gamma$). Note that this root appears only in the stability treatment, based on the full dynamic equations (31), (35) and (27), and is absent if one uses the equilibrium system equations (34), (35) and (27). Hence, the neutral curve is given by the solution $\gamma = \gamma(\omega)$ of the following equation:

$$[\gamma^2(\Gamma - 1) - R^2 - \omega^2]\,[\gamma^2(\Gamma - 1) - \tilde{R}^2 - \omega^2] = 0. \tag{46}$$

As in the case of a dry rod, all values of γ satisfying $\gamma \geq \gamma_{cr}$ correspond to the instability of a straight rod configuration, where γ_{cr} is defined as the solution of

$$\frac{\partial\gamma}{\partial\omega} = 0. \tag{47}$$

Taking the ω-derivative of equation (46) at the critical point given by equation (47), we find that

$$2\omega_{cr}^2 = \gamma^2(\Gamma^2 - 2\Gamma + 2) - R^2 - \tilde{R}^2. \tag{48}$$

Substituting this expression back into the ω-derivative of equation (46), one obtains the following quadratic equation for γ_{cr}^2:

$$0 = \Gamma^2(\Gamma - 2)\gamma_{cr}^4 - 2(\Gamma - 2)^2(R^2 + \tilde{R}^2)\,\gamma_{cr}^2 + (R^2 - \tilde{R}^2)^2, \tag{49}$$

which has a unique positive solution given by

$$\gamma_{cr}^2 = \frac{(\Gamma - 2)(R^2 + \tilde{R}^2) + \sqrt{(\Gamma - 2)^2(R^2 + \tilde{R}^2)^2 - \Gamma^2(R^2 - \tilde{R}^2)^2}}{\Gamma^2(\Gamma - 2)}. \tag{50}$$

One can rewrite equation (50) also as

$$\gamma_{cr}^2 = \frac{\left(\sqrt{(1 - \Gamma)R + \tilde{R}} + \sqrt{(1 - \Gamma)\tilde{R} + R}\right)^2}{\Gamma^2(2 - \Gamma)}, \tag{51}$$

yielding

$$\gamma_{\rm cr} = \pm\frac{\left(\sqrt{(1-\Gamma)R+\tilde{R}}+\sqrt{(1-\Gamma)\tilde{R}+R}\right)}{\Gamma\sqrt{2-\Gamma}}. \tag{52}$$

Expressing equation (52) in terms of the original variables and using equation (33), finally leads to

$$\gamma_{\rm cr} = \pm\frac{1}{\Gamma\sqrt{2-\Gamma}}\left(\sqrt{(2-\Gamma)P^2+\frac{\mu}{2}|V|^2(\Gamma+(\Gamma-4)\cos^2 v)}\right.$$
$$\left.+\sqrt{(2-\Gamma)P^2-\frac{\mu}{2}|V|^2(\Gamma+(4-3\Gamma)\cos^2 v)}\right). \tag{53}$$

An elementary analysis of equation (53) shows that the presence of a stream tends to *destabilize* the rod, and leads to a smaller value of critical twist compared with the corresponding critical value *in vacuo*, given by equation (29). For the case when $|R^2-\tilde{R}^2| = \mu|V|^2\sin^2 v \ll |P^2|$, the following asymptotic expression for equation (53) is obtained:

$$\gamma_{\rm cr} = \pm\frac{2}{\Gamma}\left(P-\frac{\mu}{2P}|V|^2\cos^2 v\right)+\mathcal{O}\left(\frac{|V|^2}{P^2}\right). \tag{54}$$

Clearly, the maximum destabilization effect is achieved for a rod placed parallel to a uniform stream, whereas a stream in the orthogonal direction does not influence the critical bifurcation value given by the dry rod bifurcation result of equation (29).

7. LINEAR STABILITY: VISCOUS LOADING CORRECTION

In an attempt to consider the effect of the viscous loading [given by the term $(\partial\mathbf{F}^{(v)}/\partial s) = \mu C_L|V|^2(\partial\boldsymbol{\Theta}/\partial s)$], on the linear stability problem according to equation (31), one should recalculate the corresponding determinant of equation (42) which defines the neutral stability curve. For this purpose, let us consider the case of a *small* viscosity contribution. It can be shown (in a similar manner as in equations (42)–(45)) that

$$\Delta(\omega) = \det\begin{vmatrix} -2i\gamma\omega(R^2+\beta)+\eta\theta+2\gamma\Lambda & \eta(\tilde{R}^2+\beta)+2i\gamma\omega\theta+i\omega\Lambda \\ \eta(R^2+\beta)+2i\gamma\omega\theta-i\omega\Lambda & 2i\gamma\omega(\tilde{R}^2+\beta)-\eta\theta+2\gamma\Lambda \end{vmatrix} = 0, \tag{55}$$

where $\Lambda \equiv \mu C_L|V|^2$. For small Λ the root $\omega=\gamma$ and those given by equation (53) will change slightly, and hence one can keep only the terms proportional to Λ in equation (55). The most interesting qualitative new effect due to viscous drag is connected with the first root, $\omega=\gamma$. Direct calculations show that this first root is changed now to

$$\omega = \gamma+\delta\sqrt{\Lambda}+\mathcal{O}(\Lambda^2/P^2), \tag{56}$$

where $\delta=\delta(\gamma, R^2, \tilde{R}^2)$ is some parameter. Hence, a straight rod which is subject to viscous loading given by the Morison equation (15) becomes unstable for *any value* of applied twist γ and tension P^2. It will bifurcate (for a high enough ω) to a helical configuration given by

$$\mathbf{X}(s) = (s, A\cos(\delta\sqrt{\Lambda}s),\quad A\sin(\delta\sqrt{\Lambda}s)). \tag{57}$$

For small Λ (i.e. $\Lambda \ll P^2$), this helical shape is very close to the straight rod configuration. The second root, equation (54) is also slightly increased because of the viscous drag, which means that the introduction of viscosity in the "classical" buckling phenomenon has a stabilization effect.

It is important however to emphasize that an exact extension for the viscous drag term is not known. Thus, it could be that the new type of viscous instability described by equation (57), which follows from using a simplified Morison form for the viscous loading, may be unsuitable for arbitrary curved and twisted configurations.

8. SUMMARY

Exact asymptotic expressions are presented for the cross-sectional hydrodynamic loading exerted on a slender curved deformable structure embedded in an ambient nonuniform potential stream. Based on these expressions we derive the corresponding asymptotic hydroelastic equations which govern the deformation of a slender elastic cylindrical body. These equations may be further simplified in the case of a *uniform* flow-field and consist of a closed system of PDEs. We find that the inertial loading exerted on a deformable slender body due to the presence of an ambient stationary stream leads generally to a destabilization of the equilibrium shape of the structure. A possible mechanism for such a destabilization effect (also valid for nonuniform ambient flow fields) can be found in the fact that the rod is compressed by the ambient stream with the help of point pressures applied at the ends which tends to decrease the effective stretching in the rod. Correspondingly it leads to a smaller value of the critical twist. It is also demonstrated that the inclusion of viscous drag results in a new type of instability of the rod, which occurs for any twist value. This viscous destabilization effect may be attributed to the selection of an improper physical model for the viscous loading term.

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